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On dual canonical bases

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Abstract
The dual basis of the canonical basis of the modified quantized enveloping algebra is studied, in particular for type A. The construction of a basis for the coordinate algebra of the $n \times n$ quantum matrices is appropriate for studying the multiplicative property. It is shown that this basis is invariant under multiplication by certain quantum minors including the quantum determinant. Then a basis of quantum $SL(n)$ is obtained by setting the quantum determinant to one. This basis turns out to be equivalent to the dual canonical basis.

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1. Introduction

Throughout this paper, the base field is $K = \mathbb{Q}(q)$, i.e., the field of quotients of polynomials in the indeterminate q with rational coefficients. Let A be an algebra over K. Two elements $b, b' \in A$ are called *equivalent* (denoted by $b \sim b'$) if there exists $m \in \mathbb{Z}$ such that $b' = q^m b$. Two elements b, b' are called *q-commuting* if $bb' \sim b'b$.

Let g be the Kac–Moody algebra associated with an $n \times n$ symmetrizable Cartan matrix A. Let $U_q(g)$ be the quantized enveloping algebra associated with g, with its two usual subalgebras $U_q(n^+)$ and $U_q(n^-)$ (see section 2 for details). The dual basis of the canonical basis of $U_q(n^-)$ has been widely studied in the literature. In [6], a conjecture posed by Berenstein and Zelevinsky is stated as follows: two elements b_1, b_2 of the dual canonical basis are q-commuting with each other, if and only if $b_1b_2 \sim b$ for some b in the dual canonical basis. This property of the basis is called the *multiplicative property*. By use of the Hall algebra technique, the multiplicative property of the dual canonical basis of $U_q(n^+)$ is studied in [14]. In [8], counter-examples are given for the Berenstein–Zelevinsky conjecture by finding some so-called imaginary vectors. There are many connections between the irreducible representations of Hecke algebras of A type and the multiplicative property of the dual canonical basis; see [8, 9].

Let $L(\lambda)$ be an irreducible highest weight module for $U_q(g)$ and let $L^*(\lambda)$ be its graded dual. In [10], Lusztig constructed a canonical basis of the tensor product $U(\lambda, \mu) := L(\lambda) \otimes L^*(\mu)$ which can be lifted to a canonical basis $\tilde{B}$ of the so-called modified quantized enveloping algebra $\tilde{U}_q(g)$. In this paper we will show that the module $L(\lambda) \otimes L^*(\mu)$ is absolutely indecomposable if the Kac–Moody algebra g is of affine or indefinite type. Next, we focus on the case of type A. By constructing a basis of the coordinate algebra $O_q(M(n))$ of the $n \times n$ quantum matrices, we get a basis of $O_q(SL(n))$ which turns out to be equivalent to the dual canonical basis. A pleasant aspect of this construction is that it is appropriate to study the multiplicative property of the basis.

2. Kashiwara's construction

Let g be the Kac–Moody algebra associated with an $n \times n$ symmetrizable Cartan matrix A. One can choose a bilinear form such that the integral weight lattice is an even integral lattice. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\Pi^v = \{\alpha_1^v, \alpha_2^v, \ldots, \alpha_n^v\}$ be the set of simple roots and the set of simple coroots, respectively. Let $U_q(g)$ be the quantized enveloping algebra associated with g with generators $E_1, \ldots, E_n,\ F_1, \ldots, F_n,\ K_1, K_1^{-1}, \ldots, K_n, K_n^{-1}$ and the usual defining relations (see, e.g., [7]) by replacing q by q^2 because we do not want to use the square root of q later.

Let $U_q(n^+)$ (resp. $U_q(n^-)$) be the subalgebra generated by $E_1, \ldots, E_n$ (resp. $F_1, \ldots, F_n$). For any dominant weight λ, denote by $L(\lambda)$ the irreducible highest weight module over $U_q(g)$ with the highest weight λ. Denote by $L^*(\lambda)$ the graded dual of $L(\lambda)$ which is an irreducible lowest weight module with the lowest weight $-\lambda$. Let $-$ be the automorphism of the algebra $U_q(g)$ given by

$$\bar{q} = q^{-1}, \qquad \bar{E}_i = E_i, \qquad \bar{F}_i = F_i, \qquad \bar{K}_i = K_i^{-1}$$

for all i. Let v_λ (resp. v_μ^*) be a highest weight vector of $L(\lambda)$ (resp. a lowest weight vector of $L^*(\mu)$). Denote also by $-$ the linear automorphism of the module $L(\lambda)$ and of the module $L^*(\mu)$ given by

$$\overline{pv_\lambda} = \bar{p}v_\lambda, \qquad \overline{pv_\mu^*} = \bar{p}v_\mu^*$$

for $p \in U_q(g)$.

Remark 2.1. Although we use $-$ to denote several different automorphisms of various spaces, one can identify the meaning of the $-$ from the context.

In [10], Lusztig constructed a canonical basis of the tensor product $U(\lambda, \mu) := L(\lambda) \otimes L^*(\mu)$ which can be lifted to a canonical basis $\tilde{B}$ of the modified quantized enveloping algebra $\tilde{U}_q(g)$.

We will not go into detail about the canonical basis of the module $U(\lambda, \mu)$. However, we would like to show one remarkable fact about the module $U(\lambda, \mu)$. It is known that if g is of finite type, $U(\lambda, \mu)$ is finite dimensional and is indecomposable if and only if one of λ and μ is zero. However, if g is of affine or indefinite type, the situation changes dramatically.

Theorem 2.2. *If g is of affine or indefinite type, then*

$$\mathrm{End}_{U_q(g)} U(\lambda, \mu) \cong \mathbb{Q}(q).$$

Hence, $U(\lambda, \mu)$ is absolutely indecomposable.

Proof. Clearly, if λ or μ is trivial, then $U(\lambda, \mu)$ is a lowest weight module or a highest weight module and the theorem holds. Hence, we may assume that both λ and μ are nontrivial.

It is known that $U(\lambda, \mu)$ is a cyclic module and is generated by $v_\lambda \otimes v_\mu^*$. For any $\psi \in \mathrm{End}_{U_q(g)} U(\lambda, \mu)$, then $\psi(v_\lambda \otimes v_\mu^*) = u(v_\lambda \otimes v_\mu^*) \in U(\lambda, \mu)_{\lambda-\mu}$ for some $u \in U_q(g)$ which is of weight zero. If $u(v_\lambda \otimes v_\mu^*)$ is not a multiple of $v_\lambda \otimes v_\mu^*$, then

$$u(v_\lambda \otimes v_\mu^*) = s(v_\lambda \otimes v_\mu^*) + \sum_i u_i v_\lambda \otimes w_i v_\mu^*$$

where $s \in \mathbb{Q}(q), u_i \in U_q(n^-), w_i \in U_q(n^+)$, for all i, and the set $\{w_i v_\mu^*\}_i$ is linearly independent. Choose $w_k v_\lambda^*$ such that its weight is maximal among all the weights of $w_i v_\lambda^*$ for all i. Assume that $u_k v_\lambda \in L(\lambda)_\Lambda$, where Λ must be smaller than λ.

1. If the Cartan matrix A is of indefinite type, then there exists α_i^v such that $\langle \lambda - \Lambda, \alpha_i^v \rangle < 0$, i.e. $\langle \lambda, \alpha_i^v \rangle < \langle \Lambda, \alpha_i^v \rangle$ and so $F_i^{\langle \lambda, \alpha_i^v \rangle + 1} u_k v_\lambda \neq 0$. However, $F_i^{\langle \lambda, \alpha_i^v \rangle + 1} u(v_\lambda \otimes v_\mu^*) = \psi\big(F_i^{\langle \lambda, \alpha_i^v \rangle + 1}(v_\lambda \otimes v_\mu^*)\big) = 0$. On the other hand, $F_i^{\langle \lambda, \alpha_i^v \rangle + 1} u(v_\lambda \otimes v_\mu^*) = \sum_{m,j} c_{ij}^{(m)} F_i^{\langle \lambda, \alpha_i^v \rangle + 1 - m} u_j v_\lambda \otimes F_i^m w_j v_\mu^*$, where $c_{ij}^{(m)} \in \mathbb{Q}(q)$. One can easily see that $c_{ik}^{(0)} = 1$. Hence, $F_i^{\langle \lambda, \alpha_i^v \rangle + 1} u_k v_\lambda = 0$. Contradiction!
2. Now, we may assume that the Cartan matrix A is of the affine type. If there exists α_i^v such that $\langle \lambda - \Lambda, \alpha_i^v \rangle < 0$, then we can prove in the same way as above. If $\langle \lambda - \Lambda, \alpha_i^v \rangle \geqslant 0$ for all i, then we must have $\langle \lambda - \Lambda, \alpha_i^v \rangle = 0$ for all i. As there exists E_i such that $E_i u_k v_\lambda \neq 0$, we have again that $F_i^{\langle \lambda, \alpha_i^v \rangle + 1} u_k v_\lambda \neq 0$.

Therefore, $u(v_\lambda \otimes v_\mu^*)$ is a multiple of $v_\lambda \otimes v_\mu^*$ and so ψ is a scalar endomorphism. Moreover, $U(\lambda, \mu)$ is absolutely indecomposable. □

Let $U^{\mathbb{Z}}(g)$ be the integral form of the quantized enveloping algebra which is a $\mathbb{Z}[q, q^{-1}]$-subalgebra of the quantized enveloping algebra $U_q(g)$ generated by the divided powers $E_i^{(s)} := E_i^s \big/ [s]!$, $F_i^{(s)} := F_i^s \big/ [s]!$, K_i, K_i^{-1} for all i. The quantum integer is defined by $[s] = q^{2s} - q^{-2s}/q^2 - q^{-2}$ (one may refer to [4] for terminology and notation).

Denote by $U_q(g)^*$ the linear dual of the algebra $U_q(g)$. Since $U_q(g)$ is a $U_q(g)$ bi-module, $U_q(g)^*$ has an induced $U_q(g)$ bi-module structure. Let

$$A_q(g) := \{f \in U_q(g)^* \mid \text{there exists } l \geqslant 0 \text{ such that}$$
$$E_{i_1}, \ldots, E_{i_l} f = f F_{i_1}, \ldots, F_{i_l} = 0 \text{ for any } i_1, \ldots, i_l\}. \tag{2.1}$$

The quantum Peter–Weyl theorem was proved in [6].

Theorem 2.3. *As $U_q(g)$ bi-modules*

$$A_q(g) \cong \oplus_{\lambda \in P} L(\lambda) \otimes L^*(\lambda)$$

where $u \otimes v \in L(\lambda) \otimes L^*(\lambda)$ viewed as a linear function on $U_q(g)$ as follows:

$$(u \otimes v)(p) = \langle up, v \rangle, \qquad \text{for} \quad p \in U_q(g),$$

where $L(\lambda)$ and $L^*(\lambda)$ are viewed as right $U_q(g)$ module and left $U_q(g)$ module, respectively.

Let A be the subring of $\mathbb{Q}(q)$ consisting of the rational functions of q which are regular at $q = 0$. Let $-$ be the ring endomorphism of $\mathbb{Q}(q)$ sending q to q^{-1}.

Let M be an integral $U_q(g)$ module. Then

$$M = \oplus_\lambda F_i^{(n)} (\mathrm{Ker}\, E_i \cap M_\lambda).$$

We define the lower Kashiwara operators e_i, f_i of M by

$$f_i\big(F_i^{(n)} u\big) = F_i^{(n+1)} u \qquad \text{and} \qquad e_i\big(F_i^{(n)} u\big) = F_i^{(n-1)} u$$

for $u \in \mathrm{Ker}\, E_i \cap M_\lambda$.

Definition 2.4. *A pair (L, B) is called a lower crystal base of M if it satisfies the following conditions:*

(1) L is a free sub-A-module of M such that $M \cong \mathbb{Q}(q) \otimes_A L$.
(2) B is a base of the $\mathbb{Q}$-vector space L/qL.
(3) $e_i L \subset L$ and $f_i L \subset L$ for any i.
(4) $e_i B \subset B \cup \{0\}$ and $f_i B \subset B \cup \{0\}$.
(5) $L = \oplus_{\lambda \in P} L_\lambda$ and $B = \cup_{\lambda \in P} B_\lambda$, where $L_\lambda = L \cap M_\lambda$, $B_\lambda = B \cap L_\lambda / q L_\lambda$.
(6) For any $b, b' \in B$, $b' = f_i b$ if and only if $b = e_i b'$.

The upper Kashiwara operators e_i' and f_i' are defined as follows: for $u \in \operatorname{Ker} E_i \cap M_\lambda$ and $0 \leqslant n \leqslant \langle \lambda, \alpha^v \rangle$,

$$e_i'\big(F_i^{(n)} u\big) = \frac{[\langle \lambda, \alpha^v \rangle - n + 1]}{[n]} F_i^{(n-1)} u,$$

and

$$f_i'\big(F_i^{(n)} u\big) = \frac{[n+1]}{[\langle \lambda, \alpha^v \rangle - n]} F_i^{(n+1)} u.$$

We say that (L, B) is an upper crystal base if (L, B) satisfies the conditions in the definition of lower crystal base with e_i', f_i' instead of e_i, f_i.

For $\lambda \in P$, we define $\psi_M \in \operatorname{Aut} M$ by

$$\psi_M(u) = q^{-2(\lambda, \lambda)} u$$

for $u \in M_\lambda$. It is known that $\psi_M^{-1} e_i' \psi_M$ (resp. $\psi_M^{-1} f_i' \psi_M$) coincides with e_i (resp. f_i) on L/qL.

In [5], Kashiwara proved that

Lemma 2.5. *(L, B) is a lower crystal base if and only if $\psi_M(L, B)$ is an upper crystal base.*

Let $\mathcal{L}(\lambda)$ be the upper crystal lattice which is the smallest A submodule of $L(\lambda)$ containing v_λ and is stable under the action of upper Kashiwara operators. Similarly, let $\mathcal{L}^*(\lambda)$ be the upper crystal lattice which is the smallest A submodule of $L^*(\lambda)$ containing v_λ^* and is stable under the action of upper Kashiwara operators. Set

$$\mathcal{L}(A_q(g)) := \oplus_{\lambda \in P_+} \mathcal{L}(\lambda) \otimes \mathcal{L}^*(\lambda).$$

Define that

$$\langle \bar{u}, p \rangle = \overline{\langle u, \bar{p} \rangle},$$

then one can check that $\overline{u \otimes v} = \bar{u} \otimes \bar{v}$ for $u \in L(\lambda)$ and $v \in L^*(\lambda)$. Hence

$$\overline{\mathcal{L}(A_q(g))} = \oplus_{\lambda \in P_+} \overline{\mathcal{L}}(\lambda) \otimes \overline{\mathcal{L}}^*(\lambda).$$

Let

$$A_q^{\mathbb{Z}}(g) = \{ f \in A_q(g) \mid \langle f, U^{\mathbb{Z}}(g) \rangle \subset \mathbb{Z}[q, q^{-1}] \}.$$

Let $u \otimes v \in L(\lambda) \otimes L^*(\lambda)$, where u (resp. v) is a weight vector of weight λ_l (resp. λ_r). Then $u \otimes v$ is called a weight vector with left weight λ_l and right weight λ_r. An element in $A_q(g)$ is called a refined weight vector if it is a linear combination of the elements $u \otimes v$ with the same left and right weights.

Let us recall the definition of balanced triple. Let V be a vector space over $\mathbb{Q}(q)$, a B-lattice of V is a B-submodule M of V such that $V \cong \mathbb{Q}(q) \otimes_B M$. Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}[q, q^{-1}]$-lattice of V, L an A-lattice of V and $\overline{L}$ an $\overline{A}$-lattice of V. In [5], it was proved that

Lemma 2.6. *Set $E = V_{\mathbb{Z}} \cap L \cap \overline{L}$. Then the following conditions are equivalent:*

(1) $E \longrightarrow V_{\mathbb{Z}} \cap L / V_{\mathbb{Z}} \cap qL$ *is an isomorphism.*
(2) $E \longrightarrow V_{\mathbb{Z}} \cap \overline{L} / V_{\mathbb{Z}} \cap q^{-1}\overline{L}$ *is an isomorphism.*
(3) $V_{\mathbb{Z}} \cap qL \oplus V_{\mathbb{Z}} \cap \overline{L} \longrightarrow V_{\mathbb{Z}}$ *is an isomorphism.*
(4) $A \otimes E \longrightarrow L, \overline{A} \otimes E \longrightarrow \overline{L}, \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} E \longrightarrow V_{\mathbb{Z}}, \mathbb{Q}(q) \otimes_{\mathbb{Z}} E \longrightarrow V$ *are isomorphisms.*

We call $(L, \overline{L}, V_{\mathbb{Z}})$ balanced if these equivalent conditions are satisfied. Let us denote by G the inverse of the isomorphism $E \longrightarrow V_{\mathbb{Z}} \cap \overline{L} / V_{\mathbb{Z}} \cap q^{-1}\overline{L}$. If B is a base of $V_{\mathbb{Z}} \cap \overline{L} / V_{\mathbb{Z}} \cap q^{-1}\overline{L}$, then $\{G(b) | b \in B\}$ is a base of V.

In [6], it was proved that $\left(A_q^{\mathbb{Z}}(g), \mathcal{L}(A_q(g)), \overline{\mathcal{L}(A_q(g))}\right)$ is a balanced triple. Hence there is a $\mathbb{Z}$ basis B' of

$$\left(A_q^{\mathbb{Z}}(g) \cap \mathcal{L}(A_q(g)) \cap \overline{\mathcal{L}(A_q(g))}\right).$$

In [7], it was shown that B' is the dual basis of the canonical basis of the modified enveloping algebra $\tilde{U}_q(g)$ if g is of finite type.

In the following, we always assume that g is of finite type. We fix a reduced expression in the longest element of the Weyl group. Let $F_{\beta_1}, F_{\beta_2}, \ldots, F_{\beta_N}$ be the ordered root vectors given, defined according to the chosen reduced expression of the longest element in the Weyl group, where N is the length of the longest element in the Weyl group. For any $I = (i_1, i_2, \ldots, i_N) \in \mathbb{Z}_+^N$, denote by F^I the monomial $F_{\beta_1}^{(i_1)} F_{\beta_2}^{(i_2)}, \ldots, F_{\beta_N}^{(i_N)}$ which forms a PBW-type basis of the subalgebra $U_q(n^-)$. The monomial E^I is defined similarly which forms a PBW-type basis of the subalgebra $U_q(n^+)$.

Let B^- and B^+ be the canonical basis of $U_q(n^-)$ and $U_q(n^+)$, respectively. For any dominant weight λ, denoted by

$$B_\lambda^- = \{b \in B^- \mid bv_\lambda \neq 0\}$$

and

$$B_\lambda^+ = \{b' \in B^+ \mid b'v_\lambda^* \neq 0\}.$$

Note that each dual canonical basis element is a refined weight vector. Hence, we only need to consider the homogeneous part $\mathcal{L}(\lambda)_\mu \otimes \mathcal{L}^*(\lambda)_\gamma \cap \overline{\mathcal{L}(\lambda)_\mu \otimes \mathcal{L}^*(\lambda)_\gamma} \cap A_q^{\mathbb{Z}}(g)$. It is well known that bv_λ for $b \in B_\lambda^-$, form an A-base of the lower crystal lattice of $L(\lambda)$, so by lemma 2.5, $q^{-2(\mu,\mu)} bv_\lambda \in \mathcal{L}(\lambda)$. Therefore, $q^{-2(\mu,\mu)-2(\gamma,\gamma)} bv_\lambda \otimes b'v_\lambda^* \in \mathcal{L}(\lambda) \otimes \mathcal{L}(\lambda)^*$ for bv_λ of weight μ and $b'v_\lambda^*$ of weight γ. Hence

$$bv_\lambda \otimes b'v_\lambda^* = q^{2(\mu,\mu)+2(\gamma,\gamma)} q^{-2(\mu,\mu)-2(\gamma,\gamma)} bv_\lambda \otimes b'v_\lambda^* \in \mathcal{L}(\lambda) \otimes \mathcal{L}^*(\lambda).$$

Note that $bv_\lambda \otimes b'v_\lambda^*$ is $-$ invariant, it lies in the intersection of $\mathcal{L}_q(g)$, $\overline{\mathcal{L}}_q(g)$ and $A_q^{\mathbb{Z}}(g)$. Since the operators e_i, f_i and $\psi_M^{-1} e_i' \psi_M$, $\psi_M^{-1} f_i' \psi_M$ agree, respectively, after modulo q, we get that the basis B' equals

$$\left\{bv_\lambda \otimes b'v_\lambda^* \mid b \in B_\lambda^-, b' \in B_\lambda^+\right\}.$$

It is well known that any canonical basis element b in B^- is of the form

$$b = F^I + \sum_{I'} a_{I,I'} F^{I'}$$

where the coefficients $a_{I,I'} \in q\mathbb{Z}[q]$ and the element b are $-$ invariant. F^I is called the leading term of b. The canonical basis elements in B^+ have the similar form.

Let

$$C_\lambda^- = \{F^I | F^I \text{ is the leading term of an element } b \in B_\lambda^-\}$$

and let

$$C_\lambda^+ = \{E^I | E^I \text{ is the leading term of an element } b' \in B_\lambda^+\}.$$

Then $C_\lambda^- v_\lambda$ (resp. $C_\lambda^+ v_\lambda^*$) is the PBW basis of $L(\lambda)$ (resp. $L(\lambda)^*$) with an order given by the chosen reduced expression of the longest element in the Weyl group.

We order the PBW-type basis $\bigcup_\lambda C_\lambda^- \otimes C_\lambda^+$ by lexicographic ordering.

Theorem 2.7. *The basis B' is characterized by the following two conditions:*

(1) $b' = F^I v_\lambda \otimes E^{I'} v_\lambda^* + \sum_{I_k, I'_K} a_{I,I'}^{I_k,I'_k} F^{I_k} v_\lambda \otimes E^{I'_k} v_\lambda^*$ where $a_{I,I'}^{I_k,I'_k} \in q\mathbb{Z}[q]$ and $a_{I,I'}^{I_k,I'_k} \neq 0$ only if $(I_k, I'_k) \leqslant (I, I')$, for any $b' \in B'$.

(2) $\overline{b'} = b'$.

Proof. Clearly, each element $bv_\lambda \otimes b'v_\lambda^*$ satisfies the two conditions. The uniqueness can be proved in the same way as in [1]. □

The following result was proved in [6].

Proposition 2.8. *Let x and y be refined weight vectors of weights (λ_l, λ_r) and (μ_l, μ_r), respectively. Then*

$$\overline{xy} = q^{2(\lambda_r,\mu_r)-2(\lambda_l,\mu_l)}\bar{y}\bar{x}.$$

By using the above proposition, one can easily verify that

Lemma 2.9. *The mapping*

$$\begin{aligned} \phi : A_q(g) &\longrightarrow A_q(g), \\ q &\mapsto q^{-1}, \\ u \otimes v &\mapsto q^{((\lambda_l,\lambda_l)-(\lambda_r,\lambda_r))}\bar{u} \otimes \bar{v}, \end{aligned} \tag{2.2}$$

if $u \otimes v$ is of the left weight λ_l and right weight λ_r, extends to an algebra anti-automorphism of the algebra $A_q(g)$ over $\mathbb{Q}$.

Let $b' \in B'$ with weights (λ_l, λ_r). Then the element $b = q^{\frac{1}{2}((\lambda_l,\lambda_l)-(\lambda_r,\lambda_r))}b'$ is invariant under the anti-automorphism ϕ. Let

$$L^* = \{b | b' \in B'\}.$$

Then L^* is also a $\mathbb{Z}[q, q^{-1}]$ basis of $A^{\mathbb{Z}}(g)$.

It is clear that the multiplicative properties of B' and L^* are the same.

Proposition 2.10. *Let $b_1, b_2 \in L^*$. Assume that $b_1b_2 \sim b$ for some $b \in L^*$. Then $b_1b_2 \sim b_2b_1$.*

Proof. Assume that $b_1b_2 = q^a b$ for some $a \in \mathbb{Z}$. Applying the anti-automorphism ϕ, we deduce that $b_2b_1 = q^{-a}b$. Hence, $b_1b_2 = q^{2a}b_2b_1$. □

3. The construction of the basis of $O_q(M(n))$

The coordinate algebra $O_q(M(n))$ of the quantum matrix is an associative algebra, generated by elements Z_{ij}, $i, j = 1, 2, \ldots, n$, subject to the following defining relations:

$$Z_{ij}Z_{ik} = q^2 Z_{ik}Z_{ij} \qquad \text{if} \quad j < k, \tag{3.1}$$

$$Z_{ij}Z_{kj} = q^2 Z_{kj}Z_{ij} \qquad \text{if} \quad i < k, \tag{3.2}$$

$$Z_{ij}Z_{st} = Z_{st}Z_{ij} \qquad \text{if} \quad i > s, j < t, \tag{3.3}$$

$$Z_{ij}Z_{st} = Z_{st}Z_{ij} + (q^2 - q^{-2})Z_{it}Z_{sj} \qquad \text{if} \quad i < s, j < t. \tag{3.4}$$

For any matrix $A = (a_{ij})_{i,j=1}^n \in M_n(\mathbb{Z}_+)$ ($\mathbb{Z}_+ = \{0, 1, \ldots\}$) we define a monomial Z^A by

$$Z^A = \prod_{i,j=1}^n Z_{ij}^{a_{ij}}, \tag{3.5}$$

where the factors are arranged in lexicographic order on $I(n) = \{(i, j) \mid i, j = 1, \ldots, n\}$. It is well known that the set $\{Z^A | A \in M_n(\mathbb{Z}_+)\}$ is a basis of the algebra $O_q(M(n))$.

From the defining relations (3.1) of the algebra $O_q(M(n))$, it is easy to show the following lemma.

Lemma 3.1.

(1) The mapping

$$^{-} : Z_{ij} \mapsto Z_{ij} \qquad q \mapsto q^{-1} \tag{3.6}$$

extends to an algebra anti-automorphism of the algebra $O_q(M(n))$ as an algebra over $\mathbb{Q}$.

(2) The mapping

$$\sigma : Z_{ij} \mapsto Z_{ji} \tag{3.7}$$

extends to an algebra automorphism of the algebra $O_q(M(n))$ as an algebra over $K = \mathbb{Q}(q)$.

For any $A = (a_{ij})_{n\times n} \in M_n(\mathbb{Z}_+)$. Let

$$\text{ro}(A) = \left(\sum_j a_{1j}, \ldots, \sum_j a_{nj}\right) = (r_1, r_2, \ldots, r_n)$$

which is called the row sum of A and

$$\text{co}(A) = \left(\sum_j a_{j1}, \ldots, \sum_j a_{jn}\right) = (c_1, c_2, \ldots, c_n)$$

which is called the column sum of A.

For any matrix $A = (a_{ij})_{i,j=1}^n \in M_n(\mathbb{Z}_+)$, a monomial having the factors of Z^A in arbitrary order. Then its expansion in terms of monomials Z^B only involves terms where $\text{ro}(B) = \text{ro}(A)$ and $\text{co}(B) = \text{co}(A)$. Let $Pr(A, s, t) = \sum_{i\leqslant s, j\leqslant t} a_{ij}$. Then $Pr(A, s, t) \geqslant Pr(B, s, t)$ for any $s, t \leqslant n$ and matrix B appeared in the expansion considered above.

From the defining relations (3.1) of the algebra $O_q(M(n))$, we have

$$\overline{Z^A} = E(A)Z^A + \sum_B c_B(A)Z^B, \tag{3.8}$$

where

$$E(A) = q^{-2(\sum_i \sum_{j>k} a_{ij}a_{ik} + \sum_i \sum_{j>k} a_{ji}a_{ki})}$$

and $B < A$, $\mathrm{ro}(B) = \mathrm{ro}(A)$, $\mathrm{co}(B) = \mathrm{co}(A)$, $c_B(A) \in \mathbb{Z}[q, q^{-1}]$, $\leqslant$ is the lexicographic ordering.

For a pair of vectors $R, C \in \mathbb{Z}_+^n$, denote by $M(R, C)$ the subspace of $O_q(M(n))$ spanned by Z^A with $\mathrm{ro}(A) = R$, and $\mathrm{co}(A) = C$. Note that $M(R, C)$ is $^-$ invariant and $O_q(M(n)) = \oplus_{R,C} M(R, C)$.

Let $D(A) = q^{-\sum_i \sum_{j>k} a_{ij}a_{ik} - \sum_i \sum_{j>k} a_{ji}a_{ki}}$ and let $Z(A) = D(A)Z^A$. Set

$$L^* = \oplus_{A \in M_n(\mathbb{Z}_+)} \mathbb{Z}[q] Z(A).$$

Theorem 3.2. *There is a unique basis* $B^* = \{b(A) | A \in M_n(\mathbb{Z}_+)\}$ *of* L^* *determined by the following conditions:*

(1) $\overline{b(A)} = b(A)$ *for all* A.
(2) $b(A) = Z(A) + \sum_{B<A} h_B(A) Z(B)$ *where* $h_B(A) \in q\mathbb{Z}[q]$ *and* $\mathrm{ro}(B) = \mathrm{ro}(A)$, $\mathrm{co}(B) = \mathrm{co}(A)$.

Proof. We rewrite equation (3.8) in terms of $Z(A)$, then

$$\overline{Z(A)} = \sum_B a_{AB} Z(B), \tag{3.9}$$

where $a_{AA} = 1$, $a_{AB} \in \mathbb{Z}[q, q^{-1}]$ and $a_{AB} = 0$ unless $B \leqslant A$, where $\leqslant$ is the lexicographic ordering. By theorem 1.2 of [1], there is an IC-basis with respect to the triple $(\{Z^A \mid A \in M_n(\mathbb{Z}_+)\}, ^-, \leqslant)$ determined by the relation stated in the context of the theorem. □

The quantum determinant $\det_q$ is defined as follows:

$$\det_q = \Sigma_{\sigma \in S_n} (-q^2)^{l(\sigma)} Z_{1\sigma(1)} Z_{2\sigma(2)}, \ldots, Z_{n\sigma(n)}. \tag{3.10}$$

It is known that $\det_q$ is a central element of the algebra $O_q(M(n))$.

For later reference we now introduce some terminology. Let $m \leqslant n$ be a positive integer. Given any two subsets $I = \{i_1, i_2, \ldots, i_m\}$ and $J = \{j_1, j_2, \ldots, j_m\}$ of $\{1, 2, \ldots, n\}$, each having cardinality m, it is clear that the subalgebra of $O_q(M(n))$ generated by the elements $Z_{i_r j_s}$ with $r, s = 1, 2, \ldots, m$, is isomorphic to $O_q(M(m))$, so we can talk about its determinant. Such a determinant is called a quantum minor, and will be denoted by $\det_q(I, J)$.

Let I, J be two subsets of $\{1, 2, \ldots, n\}$ with the same cardinality. Obviously, the dual canonical basis of the subalgebra generated by Z_{ij} for $i \in I, j \in J$ is a subset of the basis B^* of the algebra $O_q(M(n))$. More generally, if $(u, v) \leqslant (s, t)$, then the subalgebra $O_q(M(n))_{(s,t)}^{(u,v)}$ generated by $Z_{i,j}$, for $(u, v) \leqslant (i, j) \leqslant (s, t)$, is $^-$ invariant and one can construct a basis analogous to the construction of the basis considered in theorem 3.2, and obviously the resulting basis of $O_q(M(n))_{(s,t)}^{(u,v)}$ is a subset of the basis B^*.

Lemma 3.3. *The quantum determinant* $\det_q$ *is an element of the basis* B^*. *Furthermore, any quantum minor is also an element of the dual canonical basis.*

Proof. We only need to show that $\det_q$ is $^-$ invariant. It is well known that the centre of the algebra $O_q(M(n))$ is generated by the quantum determinant [12]. Note that

$$\begin{aligned} \overline{\det_q} Z_{ij} &= \overline{\overline{Z_{ij}} \det_q} = \overline{\det_q Z_{ij}} \\ &= Z_{ij} \overline{\det_q}, \end{aligned} \tag{3.11}$$

for any i, j. Hence, $\overline{\det_q}$ is a polynomial of $\det_q$. Therefore,

$$\overline{\det_q} = \det_q$$

by comparing the leading terms. □

Corollary 3.4. *The basis B^* is σ invariant. More precisely,*

$$\sigma(b(A)) = b(A^T),$$

for all $A \in M_n(\mathbb{Z}_+)$, where A^T is the transposition of A.

Proof. Let $b(A)$ be an element of the dual canonical basis B^* of the form given in theorem 3.2 (2). Then it follows that all the matrices B appearing in the expansion of $b(A)$ are obtained from A by a sequence of 2×2 submatrix transformations of the following form:

$$\begin{pmatrix} a_{ij} & a_{it} \\ a_{sj} & a_{st} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{ij}-1 & a_{it}+1 \\ a_{sj}+1 & a_{st}-1 \end{pmatrix}, \tag{3.12}$$

if both a_{ij} and a_{st} are positive. Hence B^T can be obtained from A^T by a sequence of the submatrix transformations of the form

$$\begin{pmatrix} a_{ji} & a_{ti} \\ a_{js} & a_{ts} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{ji}-1 & a_{ti}+1 \\ a_{js}+1 & a_{ts}-1 \end{pmatrix}. \tag{3.13}$$

Especially, $B^T \leqslant A^T$. Note that the monomials Z^{B^T} and $\sigma(Z^B)$ have the same factors but could be in different order. However, two generators Z_{ij} and Z_{st} which appear in the monomials but in different order must satisfy the third relation in (3.1). Hence, $Z^{B^T} = \sigma(Z^B)$

$$\sigma(b(A)) = Z(A^T) + \sum_B h_B(A) Z(B^T) \tag{3.14}$$

with $h_B(A) \in q\mathbb{Z}[q]$. Clearly,

$$\overline{\sigma(b(A)} = \sigma(b(A))$$

since σ and $^-$ commute with each other. □

Denote by I_n the $n \times n$ identity matrix.

Lemma 3.5. *For any $A \in M_n(\mathbb{Z}_+)$,*

$$Z(A)\det_q = Z(A+I_n) \qquad \text{mod } qL^*.$$

Proof. For $i < s$, $j < t$, we have

$$Z_{st}^m Z_{ij} = Z_{ij} Z_{st}^m + (q^{2-4m} - q^2) Z_{it} Z_{sj} Z_{st}^{m-1}.$$

Recall that

$$\det_q = \Sigma_{\sigma\in S_n} (-1)^{l(\sigma)} q^{2l(\sigma)} Z_{1\sigma(1)} Z_{2\sigma(2)}, \ldots, Z_{n\sigma(n)}.$$

When we compute $Z(A)\det_q$, we only have to deal with those coefficients of the form q^{-2a} with a a positive integer. Assume that

$$Z(A)\det_q = \sum a_B Z(B).$$

Clearly, $a_B \in \mathbb{Z}[q, q^{-1}]$ and the leading term is $Z(A + I_n)$ and the matrix B appearing in the expression has at least one nonzero entry in each row and each column. We need to compute $Z(A)(-1)^{l(\sigma)} q^{2l(\sigma)} Z_{1\sigma(1)} Z_{2\sigma(2)}, \ldots, Z_{n\sigma(n)}$, for all $\sigma \in S_n$. From the expression of the quantum determinant we see that there are four possibilities of producing coefficients of the form q^{-2a} with a a positive integer.

Case 1. $Z_{st}^m Z_{sj} = q^{-2m} Z_{sj} Z_{st}^m$ where $t > j$ but no Z_{it} behind. Then q^{2m} will be absorbed by $D(B)$ where $Z(B)$ is the resulting term.
Case 2. $Z_{st}^m Z_{it} = q^{-2m} Z_{it} Z_{st}^m$ where $s > i$ but no Z_{sj} appeared before. Then q^{2m} will be absorbed by $D(B)$ where $Z(B)$ is the resulting term.
Case 3. Both $Z_{st}^m Z_{sj} = q^{-2m} Z_{sj} Z_{st}^m$ where $t > j$ and $Z_{st}^m Z_{it} = q^{-2m} Z_{it} Z_{st}^m$ where $s > i$ happened. Then we get q^{-4m}. However, we will see that it will be cancelled by a term in the following case. To this end, we need to remember that the terms we are dealing with are from $Z(A) Z_{1\sigma(1)} Z_{2\sigma(2)}, \ldots, Z_{n\sigma(n)}$. Note that $l(\sigma(jt)) = l(\sigma) - 1$.
Case 4. $Z_{st}^m Z_{ij} = Z_{ij} Z_{st}^m + (q^{2-4m} - q^2) Z_{it} Z_{sj} Z_{st}^{m-1}$ where $s > i, t > j$. Then the coefficient q^{2-4m} will be cancelled by a term in case 3.

Hence, the coefficients a_B are all in $q\mathbb{Z}[q]$ except a_A which is 1. □

The following proposition follows directly from the above lemma.

Proposition 3.6. *The basis B^* is invariant under the multiplication of* $\det_q$. *More precisely,*

$$b(A) \det_q = b(A + I_n)$$

for all $A \in M_n(\mathbb{Z}_+)$.

By using this proposition, we can determine $b(A)$, if A is a diagonal matrix. Let $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$. We may assume that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ without loss of generality. Then

$$b(A) = \prod_{i=1}^{n} \det_{q,i}^{a_i - a_{i-1}}$$

where $\det_{q,i}$ is the quantum determinant of the subalgebra generated by Z_{st} for $s, t = i, \ldots, n$, and where we put $a_0 = 0$.

4. Some subalgebras

In this section, we study the multiplicative property of the basis B^*. Similar to the proof of proposition 2.9, we get

Lemma 4.1. *Let $b_1, b_2 \in B^*$. If $b_1 b_2 \sim b$ for some $b \in B^*$, then $b_1 b_2 \sim b_2 b_1$.*

Divide the matrix by a broken line ξ which consists of lines determined by equations $ax + by = m$ for $a, b \in \mathbb{Z}_+$ and $m \in \mathbb{N}$ (each line has a non-positive slope). Recall that $I(n) = \{(i, j) \mid i, j = 1, \ldots, n\}$. Let

$$I_1 = \{(x, y) \in I(n) \mid (x, y) \text{ is in the left upper side of the broken line } \xi\},$$

and let I_2 be the complement of I_1 in $I(n)$.

Let O_i be the subalgebra of $O_q(M(n))$ generated by Z_{xy} for $(x, y) \in I_i, i = 1, 2$. One can easily see that O_i is determined by the generators Z_{xy} and relations (3.1). Hence, the algebra O_i is closed under the bar action and therefore there is a basis B_i^* of the sub-lattice L_i^* of the $\mathbb{Z}[q]$-lattice L^* spanned by $\{Z(A) \mid A = (a_{xy}) \in M_n(\mathbb{Z}_+), a_{xy} = 0 \text{ if } (x, y) \in I_{3-i}\}$. Clearly, B_i^* is a subset of B^* which consists of the $b(A)$ for $A = (a_{xy}) \in M_n(\mathbb{Z}_+)$, $a_{xy} = 0$ if $(x, y) \in I_{3-i}$.

Write

$$A = A^+ + A^-,$$

where the entries of A^+ in the left upper side of the broken line ξ are zero and the entries of A^- in the right lower side (including the broken line ξ) are zero. Then

Theorem 4.2. *$b(A) \sim b(A^+)b(A^-)$ if and only if $b(A^+)b(A^-) \sim b(A^-)b(A^+)$.*

Proof. If $b(A) = q^a b(A^+)b(A^-)$ for some integer a, then $b(A^-)b(A^+) \sim b(A^+)b(A^-)$ by the above lemma.

For

$$b(A^+) = Z(A^+) + \sum_{B^+} a_{B^+A^+} Z(B^+),$$

and

$$b(A^-) = Z(A^-) + \sum_{B^-} a_{B^-A^-} Z(B^-),$$

where $a_{B^+A^+}, a_{B^-A^-} \in q\mathbb{Z}[q]$. Assume that $b(A^+)b(A^-) = q^a b(A^-)b(A^+)$, for some integer a which can be computed by only considering the leading terms. From the defining relations (3.1), the integer a must be even, say, $a = 2m$. Then $q^{-m} b(A^+)b(A^-)$ is bar-invariant with leading term $Z(A)$. Note that the coefficients we encounter only depend on the row sums and column sums. Actually, $m = \sum_j \left(r_j^+ r_j^- + c_j^+ c_j^-\right)$ where $\left(r_1^+, \ldots, r_n^+\right)$ and $\left(c_1^+, \ldots, c_n^+\right)$ (resp. $(r_1^-, \ldots, r_n^-)$ and $(c_1^-, \ldots, c_n^-)$ are the row sum and column sum, respectively, of A^+ (resp. of A^-). Then all terms produce the same m. Therefore,

$$b(A) = q^{-m} b(A^+)b(A^-)$$

by theorem 3.2. □

5. Some quantum minors

Let $\det_q(t) = \det_q(\{1, \ldots, t\}, \{n - t + 1, \ldots, n\})$, for $t = 1, 2, \ldots, n$.

Let $M_t^- = \{(i, j) \in \mathbb{N}^2 \mid 1 \leqslant i \leqslant t \text{ and } 1 \leqslant j \leqslant n - t\}$, $M_t^+ = \{(i, j) \in \mathbb{N}^2 \mid t + 1 \leqslant i \leqslant n \text{ and } n - t + 1 \leqslant j \leqslant n\}$, $M_t^l = \{(i, j) \in \mathbb{N}^2 \mid t + 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant n - t\}$, and $M_t^r = \{(i, j) \in \mathbb{N}^2 \mid 1 \leqslant i \leqslant t \text{ and } n - t + 1 \leqslant j \leqslant n\}$. The following result was proved in [3].

Lemma 5.1. *For any i, j, t,*

$$\begin{aligned} Z_{ij} \det_q(t) &= \det_q(t) Z_{ij} \text{ if } (i, j) \in M_t^l \cup M_t^r, \\ Z_{ij} \det_q(t) &= q^2 \det_q(t) Z_{ij} \text{ if } (i, j) \in M_t^-, \end{aligned} \tag{5.1}$$

and

$$Z_{ij} \det_q(t) = q^{-2} \det_q(t) Z_{ij} \text{ if } (i, j) \in M_t^+.$$

Let

$$E_t = \begin{pmatrix} 0 & I_t \\ 0 & 0 \end{pmatrix}$$

and let $q^{\mathbb{Z}} B^* = \{q^a b(A) \mid \text{for all } A \text{ and } a \in \mathbb{Z}\}$.

Theorem 5.2. *The set $q^{\mathbb{Z}} B^*$ is invariant under the multiplication of the quantum minors $\det_q(t)$ and $\sigma(\det_q(t))$. More precisely,*

$$b(A) \det_q(t) = q^{r_1 + \cdots + r_t - c_{n-t+1} - \cdots c_n} b(A + E_t). \tag{5.2}$$

$$b(A)\sigma(\det_q(t)) = q^{c_1+c_2+\cdots+c_t-r_{n-t+1}-\cdots-r_n} b\big(A + E_t^T\big), \tag{5.3}$$

where E_t^T is the transposition of the matrix E_t.

Proof. For any $A \in M_n(\mathbb{Z}_+)$, write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is a $t \times (n-t)$ submatrix, A_{12} is a $t \times t$ submatrix, A_{21} is a $(n-t) \times (n-t)$ submatrix and A_{22} is a $(n-t) \times t$ submatrix. Then two monomials Z^A and $Z^{A_{11}} Z^{A_{12}} Z^{A_{21}} Z^{A_{22}}$ have the same factors but could be in different order. However, the first monomial can be obtained from the second one by applying the third defining relation of the algebra $O_q(M(n))$. Hence,

$$Z^A = Z^{A_{11}} Z^{A_{12}} Z^{A_{21}} Z^{A_{22}},$$

and

$$Z(A)\det_q(t) = q^{-2\sum_{i\geqslant t+1, j\geqslant n-t+1} a_{ij}} D(A) Z^{A_{11}} Z^{A_{12}} \det_q(t) Z^{A_{21}} Z^{A_{22}}.$$

Apply the above lemma to $Z^{A_{12}} \det_q(t)$, then we get

$$Z(A_{12})\det_q(t) = Z(A_{12} + I_t) + \sum_{B_{12}} h_{B_{12}}(A_{12}) Z(B_{12}),$$

where $B_{12} \leqslant A_{12} + I_t$, B_{12} and $A_{12} + I_t$ have the same row sums and column sums. Hence

$$\begin{aligned} Z(A)\det_q(t) = {} & q^{-2\sum_{i\geqslant t+1, j\geqslant n-t+1} a_{ij}} D(A) D(A_{12})^{-1} \\ & \times D(A_{12} + I_t) D(A + E_t)^{-1} Z(A + E_t) \\ & + \sum_{B_{12}} h_{B_{12}}(A_{12}) q^{-2\sum_{i\geqslant t+1, j\geqslant n-t+1} a_{ij}} D(A) D(A_{12})^{-1} \\ & \times D(B_{12}) D\left(\begin{pmatrix} A_{11} & B_{12} \\ A_{21} & A_{22} \end{pmatrix}\right)^{-1} Z\left(\begin{pmatrix} A_{11} & B_{12} \\ A_{21} & A_{22} \end{pmatrix}\right). \end{aligned}$$

By direct computation, one can show that the dependence of

$$D(B_{12}) D\left(\begin{pmatrix} A_{11} & B_{12} \\ A_{21} & A_{22} \end{pmatrix}\right)^{-1}$$

on the matrix entries of B_{12} is only a dependence on the row and column sums.

Then one deduces that

$$Z(A)\det_q(t) = q^{r_1+\cdots+r_t-c_{n-t+1}-\cdots-c_n} \left(Z(A + E_t) + \sum_{D, D<A+E_t} c(D, A) Z(D) \right)$$

with $c(D, A) \in q\mathbb{Z}[q]$.

For a basis element $b(A)$ of the form given in theorem 3.2, we then deduce that

$$\begin{aligned} b(A)\det_q(t) = {} & q^{r_1+\cdots+r_t-c_{n-t+1}-\cdots-c_n} \left[\left(Z(A + E_t) + \sum_{D, D<A+E_t} c_D(A) Z(D) \right) \right. \\ & \left. + \sum_{B, B<A} h_B(A) \left(Z(B + E_t) + \sum_{D, D<B+E_t} c_D(B) Z(D) \right) \right] \end{aligned} \tag{5.4}$$

with $c_D(A), c_D(B) \in q\mathbb{Z}[q]$. By

$$b(A)\det_q(t) = q^{2(r_1+\cdots+r_t-c_{n-t+1}-\cdots-c_n)} \det_q(t) b(A),$$

we see that $(Z(A + E_t) + \sum_{D,D<A+E_t} c_D(A)Z(D)) + \sum_{B,B<A} h_B(A)(Z(B + E_t) + \sum_{D,D<B+E_t} c_D(B)Z(D))$ is $^-$ invariant, and it must be the basis element $b(A + E_t)$. Finally, apply the algebra automorphism σ. Then the second statement follows from corollary 3.4. □

Corollary 5.3. *Let*

$$A = \begin{pmatrix} a_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ c_2 & a_2 & b_2 & \cdots & b_{n-2} & b_{n-1} \\ c_3 & c_2 & a_3 & \cdots & b_{n-3} & b_{n-2} \\ \cdots & \cdots & \cdots & \cdots & & \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_2 & a_n \end{pmatrix}.$$

Then the basis element

$$b(A) \sim \Pi_{t=1}^{n-1} \det_q(t)^{b_{n-t+1}} \sigma(\det_q(t))^{c_{n-t+1}} b(\mathrm{diag}(a_1, a_2, \ldots, a_n)).$$

Proof. Successively peel off the off-diagonals of A by (5.2) and (5.3). □

Definition 5.4. *A matrix $A = (a_{ij}) \in M_n(\mathbb{Z}_+)$ is called a ladder if $a_{ij} \geqslant a_{i+1,j+1}$ for all i, j.*

Let A be a ladder. Successively peel off the off-diagonals of A by (5.2) and (5.3), the basis element $b(A)$ is equivalent to a product of the quantum minors $\det_q(t)$ and $\sigma(\det_q(t))$ and a basis element $b\left(\begin{pmatrix} A_{n-1} & 0 \\ 0 & 0 \end{pmatrix}\right)$, where A_{n-1} is a ladder of size $n-1$. Repeatedly, the basis element $b(A)$ can be written as a product of some quantum minors which are q-commuting with each other.

6. The coincidence of two bases

Let g be the finite-dimensional simple Lie algebra of type A_{n-1} and let $\Lambda_1, \ldots, \Lambda_{n-1}$ be the fundamental dominant weights. For any dominant weight λ, the irreducible highest weight module $L(\lambda)$ occurs as a sub-quotient of a suitable power of the natural representation $L(\Lambda_1)$. The simple modules $L(\Lambda_1)$ and $L(\Lambda_{n-1})$ are dual to each other and are of dimension n. Let $e_1, e_2, \ldots, e_n$ be the standard basis of $L(\Lambda_1)$ and let $e_1^*, e_2^*, \ldots, e_n^*$ be the dual basis of $L(\Lambda_{n-1})$. Then it is well known that the matrix coefficients $X_{ij} = e_i^* \otimes e_j$ satisfy the following relations:

$$X_{ij}X_{ik} = q^2 X_{ik}X_{ij} \qquad \text{if} \quad j < k, \tag{6.1}$$

$$X_{ij}X_{kj} = q^2 X_{kj}X_{ij} \qquad \text{if} \quad i < k, \tag{6.2}$$

$$X_{ij}X_{st} = X_{st}X_{ij} \qquad \text{if} \quad i > s,\ j < t, \tag{6.3}$$

$$X_{ij}X_{st} = X_{st}X_{ij} + (q^2 - q^{-2})X_{it}X_{sj} \qquad \text{if} \quad i < s,\ j < t,$$

$$\Sigma_{\sigma \in S_n} (-q^2)^{l(\sigma)} X_{1\sigma(1)} X_{2\sigma(2)}, \ldots, X_{n\sigma(n)} = 1. \tag{6.4}$$

Since the basis B^* is invariant under the multiplication of the quantum determinant, we get a basis K^* of $O_q(SL_n)(= A_q(g))$, by setting the quantum determinant to one. Clearly, the anti-automorphism $-$ induces the anti-automorphism ϕ of $O_q(SL(n))$ (see lemma 2.9). Let $X(A)$ be the image of $Z(A)$ in $O_q(SL(n))$. Then

$$\{X(A) \mid \text{at least one zero in the diagonal}\}$$

is a basis of $O_q(SL(n))$.

Lemma 6.1. *The matrix coefficients X_{ij} are both invariant under $-$ (the bar action of $A_q(g)$) and ϕ.*

Proof. It is known that $\{e_1, e_2, \ldots, e_n\}$ (resp. $\{e_1^*, e_2^*, \ldots, e_n^*\}$) is the canonical basis of $L(\Lambda_1)$ (resp. of $L(\Lambda_{n-1})$). Therefore, e_i and e_j^* are invariant under the bar action of $L(\Lambda_1)$ and $L(\Lambda_{n-1})$, respectively. Hence, the matrix coefficients X_{ij} are $-$ invariant. Note that Λ_1 and Λ_{n-1} are minuscule dominant weights so the left weight λ_l (resp. the right weight λ_r) of X_{ij} is conjugate to Λ_1 (resp. Λ_{n-1}) under the action of the Weyl group which implies that $(\lambda_l, \lambda_l) - (\lambda_r, \lambda_r) = (\Lambda_1, \Lambda_1) - (\Lambda_{n-1}, \Lambda_{n-1}) = 0$. □

The basis K^* can be described similarly to theorem 3.2 by replacing Z_{ij} by X_{ij} and $-$ by ϕ.

Theorem 6.2. *There is a unique basis*

$$\tilde{B}^* = \{\tilde{b}(A) \mid A \in M_n(\mathbb{Z}_+), \text{ at least one zero in the diagonal}\}$$

of $\tilde{L}^* = \oplus_A \mathbb{Z}[q]X(A)$ determined by the following conditions:

(1) $\phi\tilde{b}(A) = \tilde{b}(A)$ for all A.
(2) $\tilde{b}(A) = X(A) + \sum_{B<A} h_B(A)X(B)$ where $h_B(A) \in q\mathbb{Z}[q]$ and $\mathrm{ro}(B) = \mathrm{ro}(A)$, $\mathrm{co}(B) = \mathrm{co}(A)$.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with standard orthogonal basis $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$. It is well known that the root system of type A_{n-1} is a subset of $\mathbb{R}^n$ with simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $i = 1, 2, \ldots, n-1$.

The $U_q(g)$ bi-module structure can be written down explicitly (see also [12]).

For homogeneous elements x, and y with weights (λ_l, λ_r) and (μ_l, μ_r), respectively, the left action is defined by

$$E_i X_{st} = \delta_{is} X_{s-1,t},\ F_i X_{st} = \delta_{i,s+1} X_{s+1,t},\ K_i X_{st} = q^{2(\epsilon_s,\alpha_i)} X_{st}$$

with Leibniz rule

$$E_i(xy) = E_i(x)y + q^{2(\lambda_l,\alpha_i)} x E_i(y),$$
$$F_i(xy) = xF_i(y) + q^{-2(\mu_l,\alpha_i)} F_i(x)y,$$
$$K_i(xy) = q^{2(\lambda_l+\mu_l,\alpha_i)} xy.$$

The right action is defined by

$$X_{st}E_i = \delta_{i,s+1} X_{s+1,t}, \qquad X_{st}F_i = \delta_{i,s} X_{s-1,t}, \qquad X_{st}K_i = q^{2(\epsilon_s,\alpha_i)} X_{st}$$

with Leibniz rule

$$(xy)E_i = (x)E_i y + q^{2(\lambda_r,\alpha_i)} x(y)E_i,$$
$$(xy)F_i = x(y)F_i + q^{-2(\mu_r,\alpha_i)} (x)F_i y,$$
$$(xy)K_i = q^{2(\lambda_r+\mu_r,\alpha_i)} xy.$$

Denote by the same notation the image of $\det_q(i)$ in $O_q(SL(n))$. Note that $\det_q(i)$ is annihilated by the left action of E_i for all i and by the right action of F_i for all i.

For $\lambda = m_1\Lambda_1 + m_2\Lambda_2 + \cdots + m_{n-1}\Lambda_{n-1}$, where $\Lambda_1, \Lambda_2, \ldots, \Lambda_{n-1}$ are fundamental weights. The module $L(\lambda) \otimes L^*(\lambda)$ is cyclic on $v_\lambda \otimes v_\lambda^*$ which corresponds to

$$\prod_i \det_q(i)^{m_i}$$

which is an element in the basis K^*.

Let $\mathcal{L}$ be the $\mathbb{Z}[q]$-lattice spanned by

$$\left\{q^{\frac{1}{2}((\lambda_l,\lambda_l)-(\lambda_r,\lambda_r))}bv_\lambda \otimes b'v_\lambda^*\right\}.$$

The lattice $\mathcal{L}$ is invariant under the operators $\tilde{e}_i$ which is defined by

$$\tilde{e}_i\left(q^{\frac{1}{2}((\lambda_l,\lambda_l)-(\lambda_r,\lambda_r))}bv_\lambda \otimes b'v_\lambda^*\right) = q^{(\lambda_l,\alpha_i)+1}q^{\frac{1}{2}((\lambda_l,\lambda_l)-(\lambda_r,\lambda_r))}e_i bv_\lambda \otimes b'v_\lambda^*$$

where e_i is the lower Kashiwara operators for the left action. Similarly, we define the operators $\tilde{f}_i$ as well as the operators for the right action. Clearly, the lattice $\mathcal{L}$ is invariant under the action of operators $\tilde{e}_i$, $\tilde{f}_i$ as well as the analogue operators for the right action. Applying these operators to $\prod_i \det_q(i)^{m_i}$, we see that all $X(A)$ are in the lattice $\mathcal{L}$. By the uniqueness of Lusztig's construction the bases K^* and L^* are the same.

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